Week 4

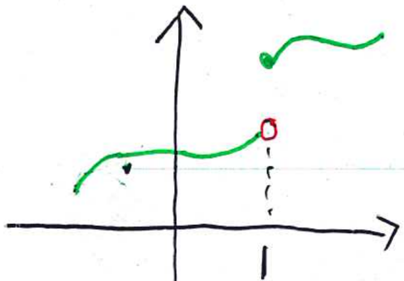
Continuity

Defn $f(x)$ is said to be continuous at a if $\boxed{\lim_{x \to a} f(x) = f(a)}$

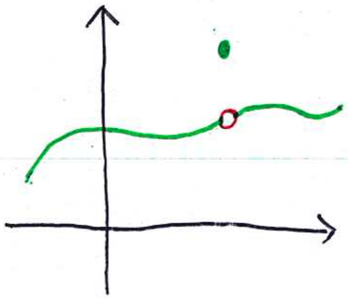
exist defined

If $A \subseteq$ Domain of f, $f(x)$ is said to be continuous on A if it is continuous at a, $\forall a \in A$

eg Discontinuous at 1

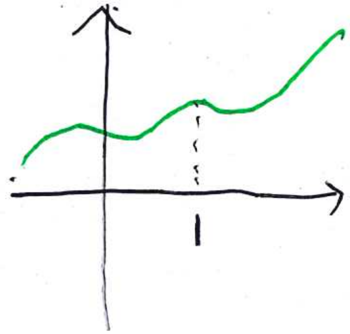


$\lim_{x \to 1} f(x)$ DNE



$\lim_{x \to 1} f(x)$ exist, but $\neq f(1)$

eg



Continuous at 1

Rmk We secretly used continuity when we found limit by substitution

eg $\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} x+1 = 1+1 = 2$

$\because$ $x+1$ is continuous

eg

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Then $f(x)$ is continuous on $\mathbb{R}$

Fact

① If f, g are continuous at a, then

$f \pm g$, fg, f/g (if $g(a) \neq 0$), f^k

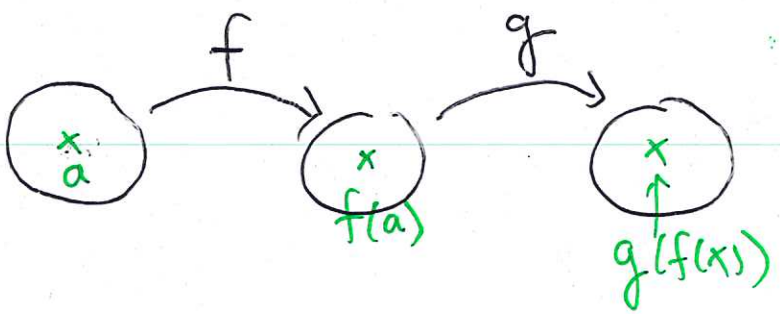
are all continuous at a

② If f is continuous at a

g is continuous at $f(a)$

then $g \circ f$ is continuous at a

Rmk $(g \circ f)(x) = g(f(x))$



Examples of Continuous function

x^a, a^x, $\log_a x$, $|x|$

$\sin x$, $\cos x$, $\tan x = \dfrac{\sin x}{\cos x}$

$\csc x = \dfrac{1}{\sin x}$, $\sec x = \dfrac{1}{\cos x}$, $\cot x = \dfrac{\cos x}{\sin x}$

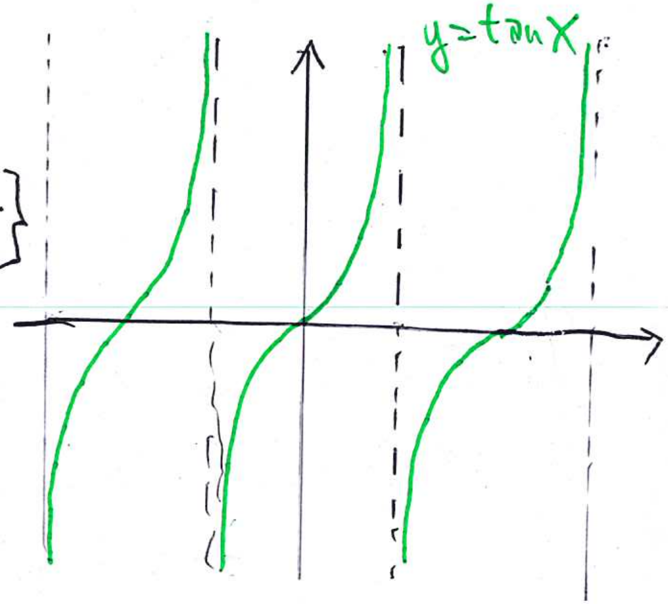
Polynomials, rational function $\dfrac{p(x)}{q(x)}$ ← polynomial

They are continuous on their domain

Domain of $\tan x$

$= \mathbb{R} \setminus \left\{ \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots \right\}$

$\tan x$ is continuous on its domain



eg $\dfrac{\sin\left(\log_4 \sqrt{x^2+1}\right)}{e^{\cos(x^2)-10|x|}}$ is also continuous

eg Find c such that $f(x)$ is continuous

(a) $f(x) = \begin{cases} \sqrt{x+9} & x \geq 0 \\ x^2+c & x<0 \end{cases}$

(b) $f(x) = \begin{cases} \dfrac{x}{|x|} & x \neq 0 \\ c & x=0 \end{cases}$

Sol (a) Clearly $f(x)$ is continuous at $x \neq 0$

At $x=0$, we need

$$f(0) = \lim_{x\to 0} f(x)$$

$$f(0) = \sqrt{0+9} = 3$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2+c = 0+c = c$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt{x+9} = \sqrt{0+9} = 3$$

$\lim_{x\to 0} f(x)$ exists $\Leftrightarrow$ $\boxed{c=3}$

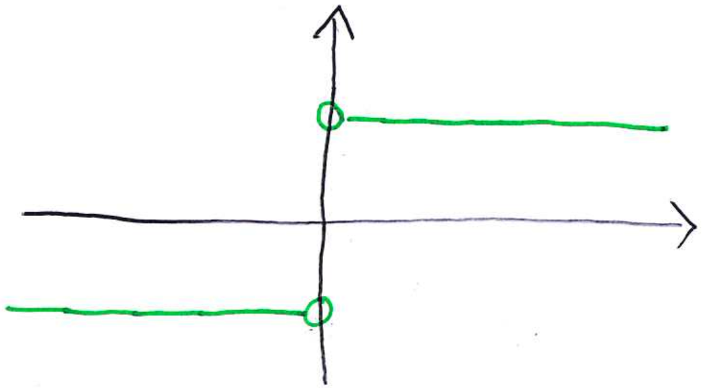
In that case, $\lim_{x\to 0} f(x) = f(0)$

$\Rightarrow$ f is continuous at 0 too if $c=3$

(b) Recall: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x<0 \end{cases}$

If $x>0$, $f(x) = \dfrac{x}{|x|} = \dfrac{x}{x} = 1$

If $x<0$, $f(x) = \dfrac{x}{|x|} = \dfrac{x}{-x} = -1$



$\lim_{x\to 0} f(x)$ DNE

$\Rightarrow f(x)$ is not continuous at 0

for any c

eg $\lim_{x\to\infty} \cos\left[\left(1+\frac{1}{2x}\right)^x\right]$

$= \cos\left(\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^x\right)$

$\because$ cos is continuous

$= \cos\left(\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^{2x\cdot\frac{1}{2}}\right)$

$= \cos\left(\sqrt{\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^{2x}}\right)$

$= \cos(\sqrt{e})$

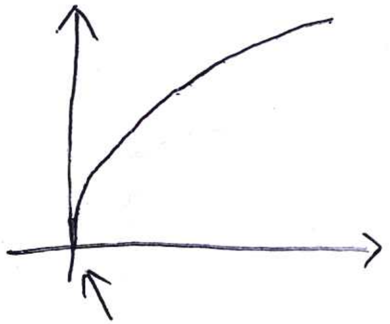
Continuous function on a closed interval $[a,b]$

means endpoints included $\{x \in \mathbb{R} : a \leq x \leq b\}$

f is said to be continous at a (at b)

$$f(a) = \lim_{x \to a^+} f(x) \quad \left(f(b) = \lim_{x \to b^-} f(x) \right)$$

e.g. $f(x) = \sqrt{x}$ Domain $= [0, \infty)$. f is continuous



f is still continuous at 0

Maximum/Minimum (Extremum)

f has absolute/global maximum at a

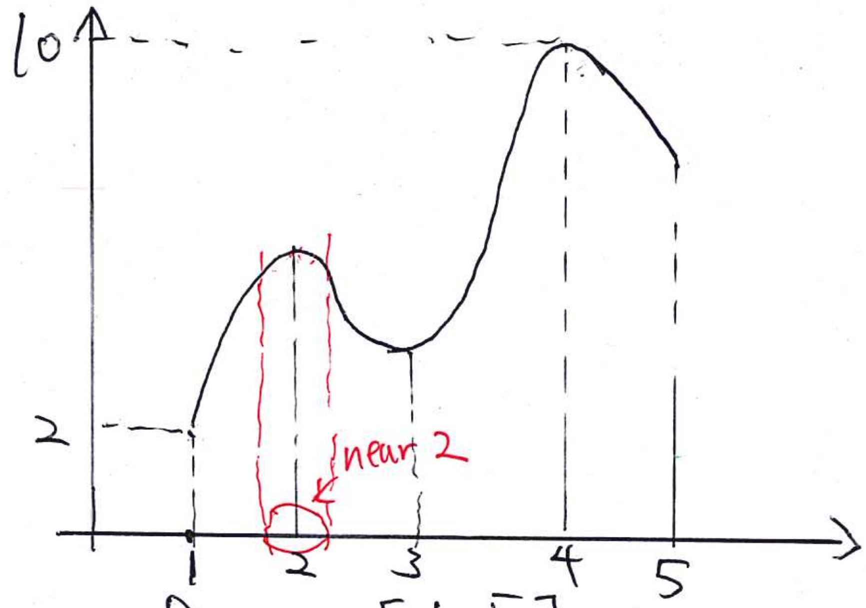
if $f(x) \leq f(a)$ $\forall x \in$ Domain of f

f has relative/local maximum at a

if $f(x) \leq f(a)$ for all x near a

Similar definitions for absolute and relative minimum.

Rmk Absolute extremum is also a relative extremum



Domain $= [1, 5]$

f has absolute maximum at 4

relative maximum at $2, 4$

absolute minimum at 1

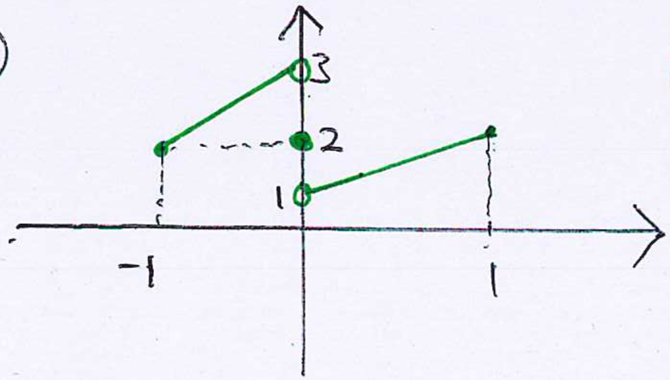
relative minimum at $1, 3, 5$

Maximum value $= 10$

Minimum value $= 2$

example of functions without max/min

(1)

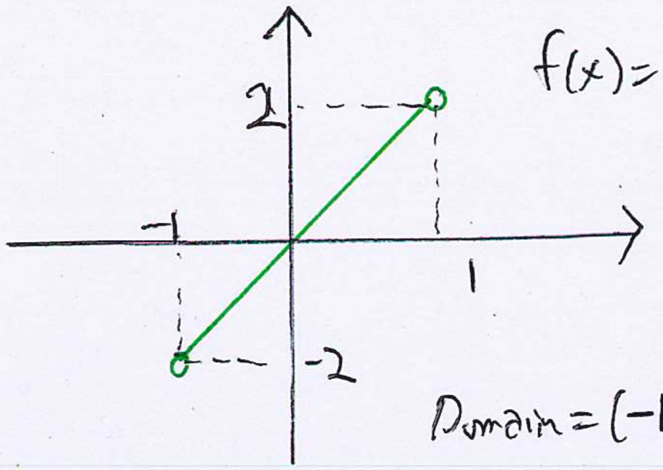


Not continuous

3 and 1 are not attainable by f

(2)

$f(x)=2x$

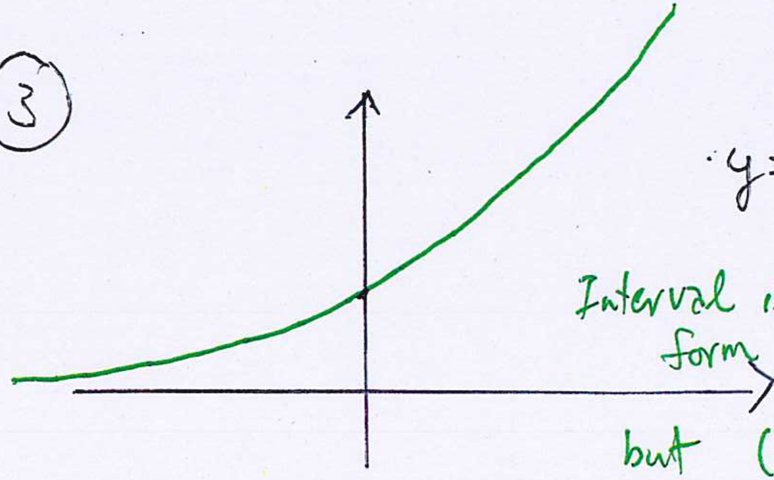


Domain $=(-1,1)$

2 and -2 are not attainable

endpoints are not included

(3)



$y=e^x$

Interval is not of the form $[a,b]$

but $(-\infty,\infty)$

Thm (Extreme Value Theorem) (EVT)

Let f be a continuous function on $[a,b]$ ← endpoints are included

then f has ... absolute maximum

and ... absolute minimum

It will be useful in the discussion of finding max/min as application of derivative

Thm (Intermediate Value theorem) (IVT)

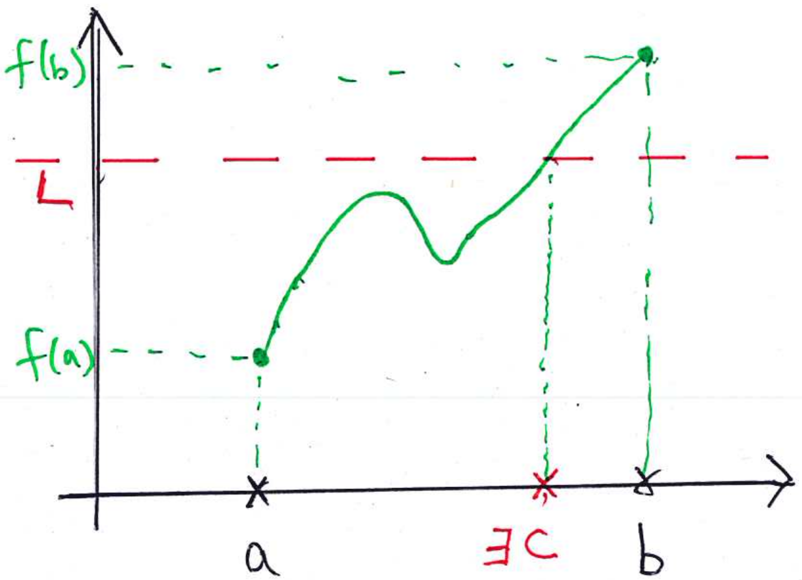
Let f be a continuous function on $[a, b]$

Suppose that $f(a) < L < f(b)$

or $f(b) < L < f(a)$, $L \in \mathbb{R}$

then $\exists c \in (a, b)$ such that

$$f(c) = L$$



eg Show that

$f(x) = x^7 + x^3 + 1$ has a real root

Sol $f(x)$ is a polynomial $\Rightarrow$ continuous

$$f(-10) < 0 < f(10)$$

IVT $\Rightarrow \exists c \in (-10, 10)$ such that

$$f(c) = 0$$

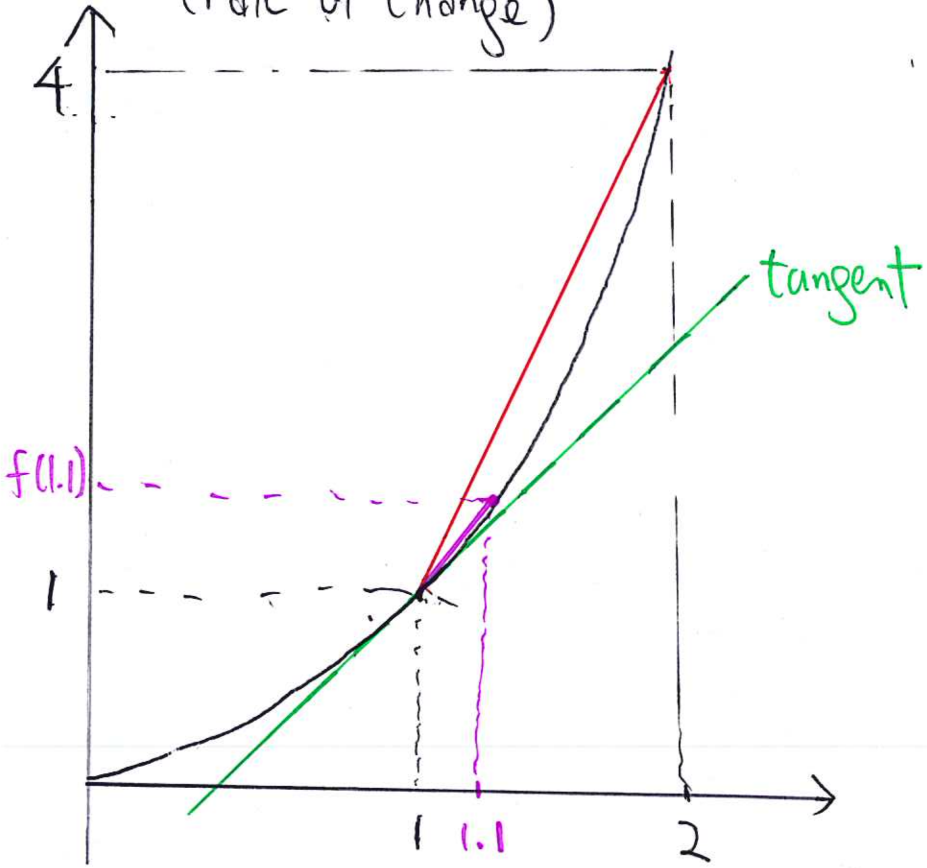
$\Rightarrow f$ has a real root c between -10 and 10

Rmk Any odd degree polynomial has at least one real root.

Differentiation / Rate of change

eg $f(x) = x^2$

Find slope of tangent at $x=1$
(rate of change)



Goal: Try to find slope of tangent

Try approximation

Slope of $= \dfrac{f(2)-f(1)}{2-1} = 3$

Slope of $= \dfrac{f(1.1)-f(1)}{1.1-1} = 2.1$

Better approximation: $\dfrac{f(1.01)-f(1)}{1.01-1} = 2.01$

$$\frac{f(0.99)-f(1)}{0.99-1} = 1.99$$

Take limit $\displaystyle\lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = 2$

slope of tangent

Here :

$$\frac{f(x)-f(a)}{x-a} = \text{Average rate of change of } f \text{ between } a \text{ and } x$$

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \text{Instantaneous rate of change of } f \text{ at } x=a$$

Rmk By a change of variable $x=a+h$ ← new variable

$x\to a \iff h\to 0$

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Def A function f is said to be differentiable at a if

$$\boxed{f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \text{ exists}}$$

$f'(a)$ is called the derivative of f at a

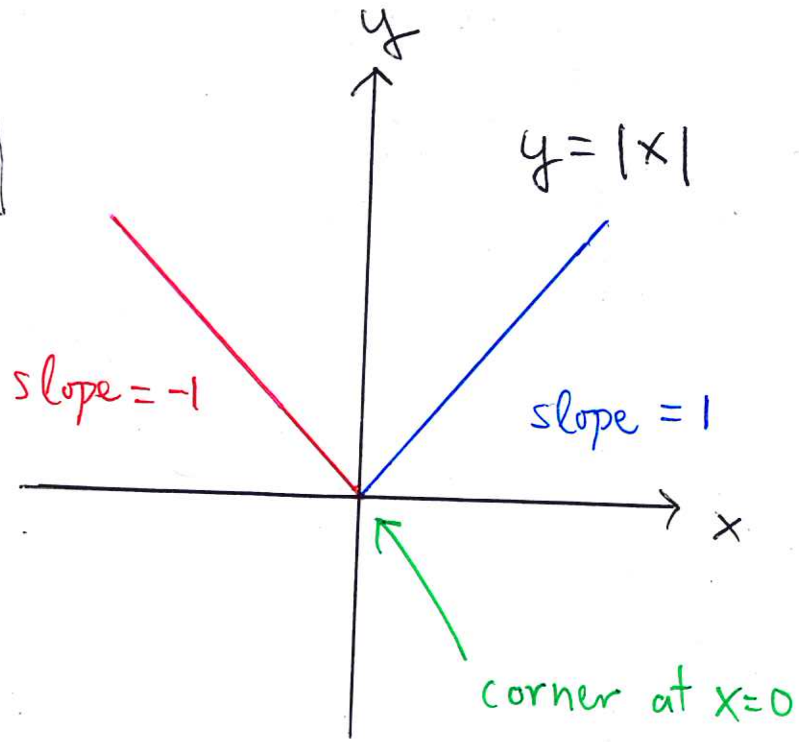
eg. $f(x) = |x|$.

Is f differentiable at 0?

$$\Updownarrow$$

Does $\lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$ exist?

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



Sol

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x - 0}{x - 0} = \boxed{1}$$

not equal

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{-x - 0}{x - 0} = \boxed{-1}$$

$\Rightarrow$ $f'(0)$ DNE $\Rightarrow$ f is not differentiable at $x = 0$

Last time: Derivative at a point

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Change of variables $x = a+h$

$$= \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

Rmk

① The derivative of f can be viewed as a function $f'(x)$ by varying a

② If domain of f is $[a, b]$, f is said to be differentiable

$$\begin{cases} \text{at } a \text{ if } \lim_{h\to 0^+} \dfrac{f(a+h)-f(a)}{h} \text{ exists} \\ \text{at } b \text{ if } \lim_{h\to 0^-} \dfrac{f(a+h)-f(a)}{h} \text{ exists} \end{cases}$$

③ Other notations: If $y = f(x)$

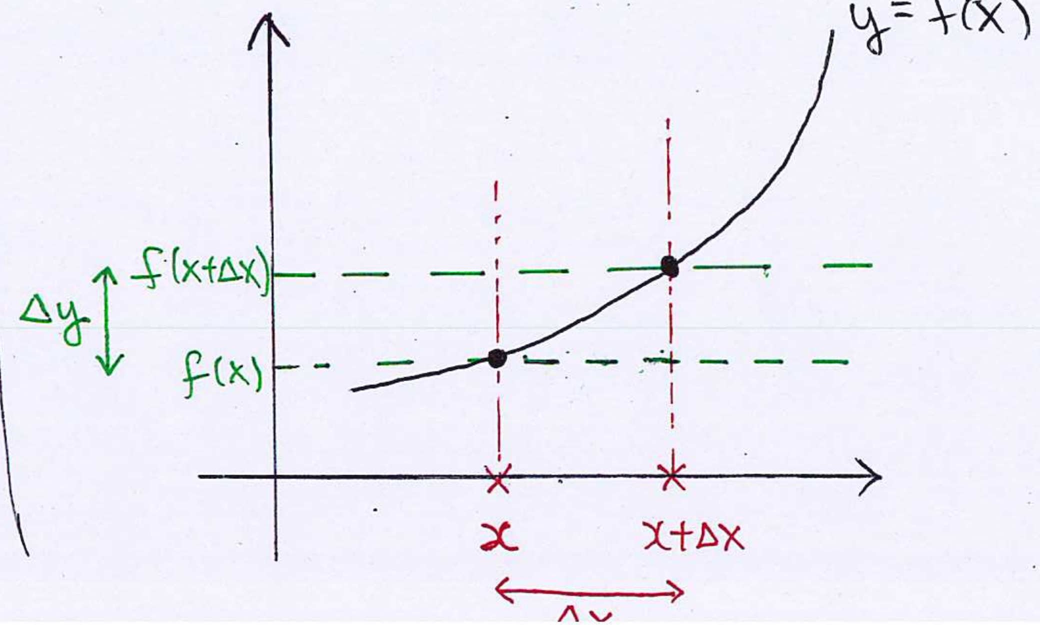
$$f'(x) = \frac{df}{dx} = \frac{dy}{dx}$$

$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a}$$

$h = \Delta x$ = difference in x

Δy = difference in $y = f(x+\Delta x) - f(x)$

$$f'(x) = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta h)-f(x)}{\Delta x}$$



eg Let $f(x) = 2x^3 + 5$. Find $f'(1)$ from definition

Sol

$$f'(1) = \lim_{x\to 1} \frac{f(x) - f(1)}{x-1}$$

$$= \lim_{x\to 1} \frac{2x^3 + 5 - 7}{x-1}$$

$$= \lim_{x\to 1} \frac{2(x^3-1)}{x-1}$$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$$= \lim_{x\to 1} \frac{2(x-1)(x^2+x+1)}{x-1}$$

$$= \lim_{x\to 1} 2(x^2+x+1)$$

$$= 2(1^2+1+1) = 6$$

eg Let $g(x) = \frac{1}{\sqrt{x}}$, $x > 0$. Find $g'(x)$

Sol

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x+h}\cdot\sqrt{x}}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\cdot\sqrt{x}} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}$$

$$= \lim_{h\to 0} \frac{x - (x+h)}{h\sqrt{x+h}\cdot\sqrt{x}\,(\sqrt{x} + \sqrt{x+h})}$$

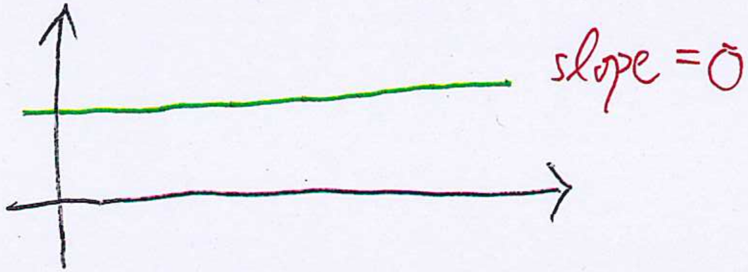
$$= \lim_{h\to 0} \frac{-1}{\sqrt{x+h}\cdot\sqrt{x}\,(\sqrt{x} + \sqrt{x+h})}$$

$$= \frac{-1}{\sqrt{x}\cdot\sqrt{x}\,(2\sqrt{x})} = -\frac{1}{2x^{\frac{3}{2}}} = -\frac{1}{2}x^{-\frac{3}{2}}$$

Proposition

① If $f(x) \equiv c$ is constant function.

$$f'(x) = \frac{d}{dx}(c) = c' = 0$$



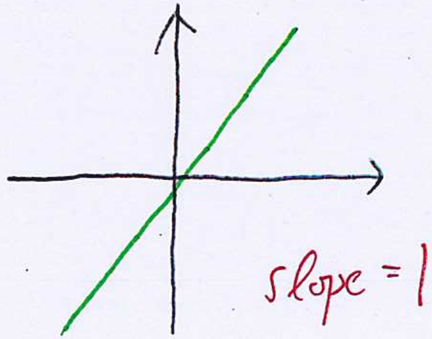
② If $f(x) = x^a$, where $a \in \mathbb{R}$ Then

$$\boxed{\text{Power Rule:}\quad f'(x) = \frac{d}{dx}(x^a) = (x^a)' = ax^{a-1}}$$

when both x^a and x^{a-1} are defined

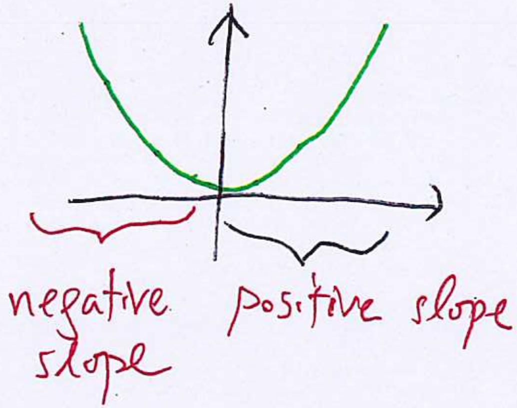
eg If $a=1$

$$\frac{d}{dx}(x) = 1$$



eg If $a=2$

$$\frac{d}{dx}(x^2) = 2x$$



Ex Prove ② when a is a positive integer

Hint: Binomial theorem or

$$a^n - b^n = (a-b)\underbrace{(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})}_{n \text{ terms}}$$

or Product Rule (Next Page)

Thm If f, g are differentiable at a.
Then $f \pm g$, fg, $\frac{f}{g}$ (if $g(a) \neq 0$), cf
are differentiable at a

c is constant

$$(f \pm g)' = f' \pm g'$$

$$(fg)' = f'g + fg' \quad \text{(Product Rule)}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \quad \text{(Quotient Rule)}$$

$$(cf)' = cf'$$

eg 1

$$\frac{d}{dx}\left[(x^3+1)(-x)\right] = \left[\frac{d}{dx}(x^3+1)\right](-x) + (x^3+1)\frac{d}{dx}(-x)$$

$$= \left[\frac{d}{dx}(x^3) + \frac{d}{dx}(1)\right](-x) + (x^3+1)(-1)$$

$$= (3x^2+0)(-x) - x^3 - 1$$

$$= -4x^3 - 1$$

eg 2

$$\frac{d}{dx}\left(\frac{x}{x^2+1}\right) = \frac{\left(\frac{d}{dx}x\right)(x^2+1) - x\left[\frac{d}{dx}(x^2+1)\right]}{(x^2+1)^2}$$

$$= \frac{(1)(x^2+1) - x(2x+0)}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(x^2+1)^2}$$

Pf of Product Rule

$$(fg)'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{[f(x+h) - f(x)]g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)[g(x+h) - g(x)]}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \boxed{\lim_{h\to 0} g(x+h)} + \lim_{h\to 0} f(x) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) \boxed{g(x)} + f(x)g'(x)$$

Rmk g $\underbrace{\text{differentiable} \Rightarrow \text{continuous}}_{\text{Prove later}}$ $\Rightarrow \lim_{h\to 0} g(x+h) = g(x+0) = g(x)$